

Remarks on Fractional Discrete Cone Control Systems with n -Orders and Their Stability

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Abstract In the paper, fractional discrete cone control systems with n -orders are considered. Some relations between invariance and (asymptotic) stability properties of the presented systems are discussed. Operators employed to the considered systems are Caputo-, Riemman-Louville-, and Grünwald-Letnikov type ones. Cone systems with control, which are particular invariant systems with control, together with their stability and asymptotic stability properties are examined.

Keywords Fractional order discrete systems · Invariance · Stability · Asymptotic stability

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1 Introduction

Positive systems are the systems whose state and input variables are never negative, with a given nonnegative initial state. These systems appear frequently in practical applications as well as in real phenomena, among other in biology, medicine, economics, electrotechnics, control system design, etc. (see [6, 7, 13, 25, 26] and the references therein). The natural generalization of positive systems are cone systems, i.e., systems whose trajectories always remain in the given cone if they are initialized in this cone. Moreover, the special attention can be put on invariant cone systems, since they can be employed to design stabilizing controllers.

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In recent years, there has been observed a growing interest in the theory and applications of fractional differential and difference equations. Many authors proved that such types of equations are more adequate for modeling physical and chemical processes than equations with integer order. Fractional differential and difference equations describe many phenomena arising in engineering, physics, or economics. In fact, one can find several applications in viscoelasticity, electrochemistry, electromagnetic, etc. For example, Machado in [19] gave a novel method for the design of fractional order digital controllers.

In both theory and applications, one can meet several definitions of the fractional derivatives, among which the most popular are Caputo-, Riemman-Liouville-, and Grünwald-Letnikov operators, so there appears the problem how to deal with differences resulting from the application of these operators. The first steps in this topic were made in [20]. Properties of the fractional sum, Caputo- and Riemman-Liouville-type difference operators, were developed in [1–4, 21]. Basic information on fractional calculus concept, ideas, and applications of these operators can be found for example in [15, 18, 24]. In [8], there was adopted a more general fractional h -difference Riemman-Liouville operator, where on the one hand h represents sample step, on the other hand, for h tending to zero, the solutions of the fractional difference equation may be seen as approximations of the solutions of corresponding Riemann-Liouville fractional equations.

The goal of the paper is to examine under which conditions control systems are cone systems. To this aim, in Section 2, we present the needed notation and properties of the h -difference operators of fractional order (with arbitrary $h > 0$). Operators, which we consider are the three most important among all fractional operators: Caputo-, Riemman-Liouville-, and Grünwald-Letnikov type difference operators. In [11], it is shown that these three types of h -difference fractional operators are related to each other. Moreover, the Grünwald-Letnikov-type fractional h -difference operator can be expressed by the Riemann-Liouville-type fractional h -difference operator. So, systems with these operators can be studied simultaneously. Taking into account this fact, in Section 3, there is introduced a discrete-time cone control system with fractional order and properties of its trajectories are discussed. Since a cone is a special case of a polyhedron, in Sections 4 and 5, basing on some properties of polyhedron contractiveness, the problems of stability and asymptotic controllability of class of consider systems are tackled.

2 Some Preliminaries

For $\alpha > 0$, $h > 0$ and $a \in \mathbb{R}$ let

$$(h\mathbb{N})_a := \{a, a + h, a + 2h, \dots\}.$$

For a function $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$, then the *forward h -difference operator* is denoted by

$$(\Delta_h x)(t) := \frac{x(t+h) - x(t)}{h}, \quad t = a + kh, k \in \mathbb{N}_0,$$

while the *h -difference sum* is given by

$$({}_a\Delta_h^{-1}x)(t) := h \sum_{i=0}^k x(a + ih),$$

where $t = a + (k+1)h$ and $k \in \mathbb{N}_0$. Additionally, we define $({}_a\Delta_h^{-1}x)(a) := 0$.

For arbitrary t , $\alpha \in \mathbb{R}$ the *h -factorial function* is defined by

$$t_h^{(\alpha)} := h^\alpha \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)}, \quad (1)$$

where $\frac{t}{h} \notin \mathbb{Z}_- := \{-1, -2, -3, \dots\}$, and we use the convention that division at a pole yields zero. Notice that if we use the general binomial coefficient $\binom{a}{b} := \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$, then function (1) can be rewritten as

$$t_h^{(\alpha)} = h^\alpha \Gamma(\alpha + 1) \binom{\frac{t}{h}}{\alpha}.$$

Lemma 1 [10] *If $0 < \alpha \leq 1$, then*

$$(-1)^s \binom{\alpha}{s+1} \geq 0 \quad \text{for } s \in \mathbb{N}_1.$$

For $\alpha = 1$ one gets $(-1)^s \binom{1}{s+1} = 0$.

For a function $x \in (h\mathbb{N})_a$ the fractional h -sum of order $\alpha > 0$ is defined by

$$\begin{aligned} ({}_a\Delta_h^{-\alpha}x)(t) &:= \frac{h}{\Gamma(\alpha)} \sum_{k=0}^n (t-a-(k+1)h)^{(\alpha-1)}_h x(a+kh) \\ &= h^\alpha \sum_{k=0}^n \binom{\frac{t-a}{h}-k-1}{\alpha-1} x(a+kh), \\ ({}_a\Delta_h^0x)(t) &:= x(t) \end{aligned}$$

where $t = a + (\alpha + n)h$. If $\psi(r) = (r-a+\mu h)^{(\mu)}_h$ for any $r \in (h\mathbb{N})_a$, then (see [8])

$$({}_a\Delta_h^{-\alpha}\psi)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a+\mu h)^{(\mu+\alpha)}_h, \quad (2)$$

where $t \in (h\mathbb{N})_{a+\alpha h}$. Using the general binomial coefficient, one can rewrite Eq. (2) in the form

$$({}_a\Delta_h^{-\alpha}\psi)(t) = \Gamma(\mu+1) \binom{n+\alpha+\mu}{n} h^{\mu+\alpha}.$$

Note that if $\psi \equiv 1$, then for $\mu = 0$, $a = (1-\alpha)h$, and $t = nh + a + \alpha h$, it holds

$$({}_a\Delta_h^{-\alpha}1)(t) = \frac{1}{\Gamma(\alpha+1)} (t-a)^{(\alpha)}_h = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} h^\alpha = \binom{n+\alpha}{n} h^\alpha.$$

Definition 2 [5] Let $\alpha \in (0, 1]$ and $a \in \mathbb{R}$. The Riemann-Liouville-type fractional h -difference operator ${}^{RL}_a\Delta_h^\alpha$ of order α for a function $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ is defined by

$$\left({}^{RL}_a\Delta_h^\alpha x\right)(t) = \left(\Delta_h \left({}_a\Delta_h^{-(1-\alpha)}x\right)\right)(t),$$

where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$.

Definition 3 [21] Let $\alpha \in (0, 1]$ and $a \in \mathbb{R}$. The Caputo-type fractional h -difference operator ${}^C_a\Delta_h^\alpha$ of order α for a function $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ is defined by

$$\left({}^C_a\Delta_h^\alpha x\right)(t) = \left({}_a\Delta_h^{-(1-\alpha)}(\Delta_h x)\right)(t),$$

where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$.

Note that the operator ${}^C_a\Delta_h^\alpha$ for any $\alpha \in (0, 1]$ changes the domain of the function x , i.e., it maps real valued functions defined on the set $(h\mathbb{N})_a$ into real valued functions defined on the set $(h\mathbb{N})_{a+(1-\alpha)h}$. Moreover, for $\alpha = 1$, we have $\left({}^C_a\Delta_h^1 x\right)(t) = (\Delta_h x)(t)$. Similarly, it

holds also for the Riemann-Liouville-type h -difference operator. Moreover, for $\alpha \in (0, 1]$, it holds (see [11]):

$$\begin{aligned} \left({}^C_a \Delta_h^\alpha x\right)(t) &= \left({}^{RL}_a \Delta_h^\alpha x\right)(t) - \frac{x(a) \cdot (t-a)_h^{(-\alpha)}}{\Gamma(1-\alpha)} \\ &= \left({}^{RL}_a \Delta_h^\alpha x\right)(t) - \frac{x(a)}{h^\alpha} \left(\frac{t-a}{h}\right)_{-\alpha}, \end{aligned}$$

for $t \in (h\mathbb{N})_{a+(1-\alpha)h}$.

Let us recall that the \mathcal{Z} -transform of a sequence $\{y(n)\}_{n \in \mathbb{N}_0}$ is a complex function $Y(z)$ given by

$$Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} y(k)z^{-k},$$

where $z \in \mathbb{C}$ is a complex variable for which the series $\sum_{k=0}^{\infty} y(k)z^{-k}$ converges absolutely.

Proposition 4 [22] For $a \in \mathbb{R}$, $\alpha \in (0, 1]$, let us define $y(n) := \left({}^{RL}_a \Delta_h^\alpha x\right)(t)$, where $t = a + (q - \alpha)h + nh$ and $t_0 = a + (q - \alpha)h$. Then

$$\mathcal{Z}[y](z) = h^{-\alpha} \left(z^q \left(\frac{z}{z-1}\right)^{-\alpha} X(z) - z \sum_{k=0}^{q-1} (z-1)^{q-k-1} \left(\Delta_h^k \left({}_a \Delta^{-(q-\alpha)} x\right)\right)(t_0)\right),$$

where $X(z) = \mathcal{Z}[\bar{x}](z)$ and $\bar{x}(n) := x(a + nh)$.

Proposition 5 [22] For $a \in \mathbb{R}$, $\alpha \in (0, 1]$, $q \in \mathbb{N}_1$ let us define $y(n) := \left({}^C_a \Delta_h^\alpha x\right)(t)$, where $t = a + (q - \alpha)h + nh$ and $t_0 = a + (q - \alpha)h$. Then

$$\mathcal{Z}[y](z) = h^{-\alpha} z^q \left(\frac{z}{z-1}\right)^{-\alpha} \cdot \left(X(z) - \frac{z}{z-1} \sum_{k=0}^{q-1} (z-1)^{-k} \left(\Delta_h^k x\right)(a)\right)$$

where $X(z) = \mathcal{Z}[\bar{x}](z)$ and $\bar{x}(n) := x(a + nh)$.

The last operator that we take under our consideration is the fractional h -difference Grünwald-Letnikov-type operator.

Definition 6 [11] Let $\alpha \in \mathbb{R}$. The Grünwald-Letnikov-type h -difference operator ${}^{GL}_a \Delta_h^\alpha$ of order α for a function $x : (h\mathbb{N})_a \rightarrow \mathbb{R}$ is defined by

$$\left({}^{GL}_a \Delta_h^\alpha x\right)(t) = \sum_{s=0}^{\frac{t-a}{h}} c_s^{(\alpha)} x(t - sh)$$

where

$$c_s^{(\alpha)} = (-1)^s \binom{\alpha}{s} \frac{1}{h^\alpha}$$

with

$$\binom{\alpha}{s} = \begin{cases} 1 & \text{for } s = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-s+1)}{s!} & \text{for } s \in \mathbb{N}. \end{cases}$$

If $a = (\alpha - 1)h$ and $x(t) := y(t - a)$ for any $t \in (h\mathbb{N})_a$ then, see [11]

$$\left({}^{GL}_0 \Delta_h^\alpha y\right)(t) = \left({}^{RL}_a \Delta_h^\alpha x\right)(t - h) \quad (3)$$

Proposition 7 [22] For $t = a + \alpha h + kh \in (h\mathbb{Z})_{a+\alpha h}$ let us denote $y(k) := ({}_a^{GL} \Delta_h^{-\alpha} x)(t)$ and $\bar{x}(k) := x(a + kh)$. Then

$$\mathcal{Z}[y](z) = \left(\frac{hz}{z-1} \right)^\alpha X(z),$$

where $X(z) := \mathcal{Z}[\bar{x}](z)$.

Since the Grünwald-Letnikov-type h -difference operator can be expressed by the Riemann-Liouville-type fractional h -difference operator (see Eq. 3), we restrict our consideration only to the Caputo- and Riemann-Liouville-type fractional h -difference operators.

3 Cone Systems

In order to define cone systems, let us discuss first the problem of existing of solutions of a nonlinear control system of fractional order. The reasoning is similar to the one given for finding solutions of the nonlinear autonomous system of the fractional order $\alpha_i \in (0, 1]$ given in [23], so we present only main steps of it.

Let $i = 1, \dots, n$ and $0 < \alpha_i \leq 1$. Let us consider the following fractional Caputo h -difference system with n orders $\alpha_1, \dots, \alpha_n$ as follows

$$\left({}_{t_{0i}}^C \Delta_h^{\alpha_i} x_i \right)(t) = f_i(t, x_1(a_1 + t), x_2(a_2 + t), \dots, x_n(a_n + t), u(t)), \quad (4)$$

with initial values

$$x_i(t_{0i}) = x_{0i} \in \mathbb{R}, \quad (5)$$

where $a_i = (\alpha_i - 1)h \in (-h, 0]$, $t_{0i} = a_i + n_0 h \in (h\mathbb{N})_{a_i}$, $n_0 \in \mathbb{N}_0$, $t \in (h\mathbb{N})_{n_0 h}$, $f_i : (h\mathbb{N})_0 \times \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}$, $i = 1, \dots, n$, $\mathcal{U} \subseteq \mathbb{R}^m$. The set \mathcal{U} is called the control space and satisfies the following property: \mathcal{U} is such that $\mathcal{U} \subseteq \text{int}\mathcal{U}$ and any two points in the same connected component of \mathcal{U} can be jointed by a smooth curve lying in $\text{int}\mathcal{U}$, except for end points. Let $J_0(m)$ denotes the set of all sequences $u = (u_0, u_1, \dots)$, where $u_n := u(nh) \in \mathcal{U}$, $n \in \mathbb{N}_0$. We assume that function f depends on finite number of elements u_i .

Note that if the Riemann-Liouville-type fractional h -difference operator ${}_{t_{0i}}^{RL} \Delta_h^{\alpha_i}$ is used instead of the Caputo-type h -difference operator ${}_{t_{0i}}^C \Delta_h^{\alpha_i}$ in Eq. 4, then one gets the fractional Riemann-Liouville h -difference system with n orders of the form

$$\left({}_{t_{0i}}^{RL} \Delta_h^{\alpha_i} x_i \right)(t) = f_i(t, x_1(a_1 + t), x_2(a_2 + t), \dots, x_n(a_n + t), u(t)). \quad (6)$$

Recall the constant vector $(X^e, u^e) := (x_1^e, x_2^e, \dots, x_n^e, u^e)^T$ is an *equilibrium point* from time $t_0 = n_0 h$ of fractional difference system (4) if and only if for any $t \in (h\mathbb{N})_a$

$$\left({}_{t_{0i}}^C \Delta_h^{\alpha_i} x_i^e \right)(t) = f_i(t, x_1^e, x_2^e, \dots, x_n^e, u^e), \quad i = 1, \dots, n$$

and

$$\left({}_{t_{0i}}^{RL} \Delta_h^{\alpha_i} x_i^e \right)(t) = f_i(t, x_1^e, x_2^e, \dots, x_n^e, u^e), \quad i = 1, \dots, n$$

in the case of the Riemann-Liouville h -difference systems.

Remark 8 For the Caputo h -difference system $\left({}_{t_{0i}}^C \Delta_h^{\alpha_i} x_i^e \right)(t) \equiv 0$, the constant vector $(X^e, u^e) = (x_1^e, x_2^e, \dots, x_n^e, u^e)^T$ is an equilibrium point from time $t_0 = n_0 h$ of the Caputo

fractional h -difference system (4) if and only if $f_i(t, X^e, u^e) = 0$, $i = 1, \dots, n$ for all $t \in (h\mathbb{N})_{n_0h}$.

For simplicity, we state all definitions and theorems for the case when the equilibrium point is $(0, 0) \in \mathbb{R}^n \times \mathcal{U}$, i.e. $x_i^e = 0$, $i = 1, \dots, n$ and $u^e = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via certain change of variables.

For $k \in \mathbb{N}_{n_0}$, $n_0 \in \mathbb{N}_0$, let us define

$$\left({}^C_{n_0}\Delta^{(\alpha)}X\right)(k) := \begin{bmatrix} \left({}^C_{t_{01}}\Delta_h^{\alpha_1}x_1\right)(kh) \\ \vdots \\ \left({}^C_{t_{0n}}\Delta_h^{\alpha_n}x_n\right)(kh) \end{bmatrix}$$

or

$$\left({}^{RL}_{n_0}\Delta^{(\alpha)}X\right)(k) := \begin{bmatrix} \left({}^{RL}_{t_{01}}\Delta_h^{\alpha_1}x_1\right)(kh) \\ \vdots \\ \left({}^{RL}_{t_{0n}}\Delta_h^{\alpha_n}x_n\right)(kh) \end{bmatrix}$$

for the Caputo or Riemann-Liouville h -difference systems, respectively, and

$$F(k, X(k), u(k)) := \begin{bmatrix} f_1(kh, x_1(a_1 + kh), x_2(a_2 + kh), \dots, x_n(a_n + kh), u(kh)) \\ \vdots \\ f_n(kh, x_1(a_1 + kh), x_2(a_2 + kh), \dots, x_n(a_n + kh), u(kh)) \end{bmatrix}.$$

Then we can write systems (4) or (6) respectively, in the following forms

$$\left({}^C_{n_0}\Delta^{(\alpha)}X\right)(k) = F(k, X(k), u(k)) \quad (7a)$$

or

$$\left({}^{RL}_{n_0}\Delta^{(\alpha)}X\right)(k) = F(k, X(k), u(k)) \quad (7b)$$

where $F : \mathbb{N}_0 \times \mathbb{R}^n \times J_0(m) \rightarrow \mathbb{R}^n$. Therefore, systems (7a) and (7b) can be expressed in one compact form

$$\left({}_{n_0}\Delta^{(\alpha)}X\right)(k) = F(k, X(k), u(k)), \quad (8)$$

with the initial condition

$$X_0 := \begin{bmatrix} x_1(a_1 + n_0h) \\ \vdots \\ x_n(a_n + n_0h) \end{bmatrix} = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0n} \end{bmatrix} \in \mathbb{R}^n. \quad (9)$$

Then the solution

$$X(k) = \gamma(X_0, a + kh, u(kh)), \quad k \geq 0, \quad (10)$$

of IVP given by Eqs. 8 and 9 can be obtain using the same reasoning as in [23] and in [10]. It is a uniquely defined map $\gamma : \mathbb{R}^n \times (h\mathbb{N})_a \times J_0(m) \rightarrow \mathbb{R}^n$ by initial state X_0 and control sequence $u \in J_0(m)$ and described by

$$\begin{aligned} \gamma(X_0, a, u(0)) &= X_0 \\ \gamma(X_0, a + kh, u(kh)) &= I_k \cdot X + \\ &\quad \sum_{j=0}^k (-1)^j \cdot \Delta_j \cdot F(n_0 + k - j, \gamma(X_0, a + (k - j)h, u((k - j)h))) \end{aligned} \quad (11)$$

$$\text{where } \Lambda_j = \begin{bmatrix} \binom{-\alpha_1}{j} & 0 & \dots & 0 \\ 0 & \binom{-\alpha_2}{j} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \binom{-\alpha_n}{j} \end{bmatrix} \in \mathbb{R}^{n \times n}, \text{ and}$$

$$\mathbb{R}^{n \times n} \ni I_k = \begin{cases} \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}, & \text{for Caputo } h - \text{difference systems} \\ \begin{bmatrix} \binom{k+\alpha_1}{k+1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \binom{k+\alpha_n}{k+1} \end{bmatrix}, & \text{for Riemann-Liouville } h - \text{difference systems.} \end{cases}$$

So, $\gamma(X_0, \cdot, u)$ is defined by its values $\gamma(X_0, kh, u(kh)) = X(k)$, $k \in \mathbb{N}_0$, and denotes the state forward trajectory of system (8).

Let us take a nonsingular matrix

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

with i th, $i = 1, \dots, n$, row given as $p_i = (p_{i1}, \dots, p_{in})$. Then the set

$$K_P := \{x \in \mathbb{R}^n : \forall i = 1, \dots, n : p_i x \geq 0\} \quad (12)$$

is called a *linear cone of state generated by the matrix P in \mathbb{R}^n* (see [12, 14]). If $\mathcal{X}_n := \{X : \mathbb{N}_0 \rightarrow \mathbb{R}^n\}$, then the set

$$\mathcal{P} := \{X \in \mathcal{X}_n : X(k) \in K_P \forall k \in \mathbb{N}_0\} \quad (13)$$

is called a *linear cone of states with the vertex at 0 generated by the matrix P in the space \mathcal{X}_n* where $x_i : \mathbb{N}_{a_i} \rightarrow \mathbb{R}$.

Definition 9 Let $P \in \mathbb{R}^{n \times n}$ be a given matrix. The nonlinear fractional difference system (8) together with initial condition (9) is called a \mathcal{P} cone fractional system if $X(\cdot) \in \mathcal{P}$ for any $X_0 \in K_P$.

Theorem 10 Let K and \mathcal{P} be given as in Eqs. 12 and 13 together with $X_0 \in K_P$. If there exists a control $u \in J_0(m)$ such that for every $x \in K_P$ it holds

$$F(k, x, u) + \Lambda_{k,k} \cdot x \in K_P,$$

then for every $k \in \mathbb{N}_0$ system (4) or (6) is a \mathcal{P} cone system.

Proof The proof uses the analogous reasoning as the one given in [10] for the similar result but without a control. For the proof, we use mathematical induction. For $k = 1$, the formula holds, since if $X_0 \in K_P$, then we get $X(1) = F(0, X_0, u(0)) + \Lambda_{1,1} X_0 \in K_P$. Now, we assume that the hypothesis is true for some k , i.e., $F(k, x, u) + \Lambda_{k,k} \cdot x \in K_P$ for every

$x \in K_P$ and $u \in J_0(m)$. This means that $X(k) \in K_P$. Then, by assumption and Lemma 1, it holds

$$p_i \cdot X(k+1) = p_i \cdot (F(k, X(k), u(k)) + \Lambda_{k,k} X(k)) + \sum_{s=1}^k \Lambda_{k,s} \cdot p_i \cdot X(k-s) \geq 0.$$

Hence from mathematical induction, the thesis holds for any natural k . \square

Note that for system (8) together with initial condition (9) and with the right-hand side autonomous also the implication “only if” in Theorem 10 is true.

Corollary 11 *Let P be a nonsingular $n \times n$ matrix. Then system (8) together with initial condition (9) with the right-hand side $F(k, X(k), u(k)) = AX(k) + Bu(k)$, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, is a $(\mathcal{P}, \mathcal{Q})$ cone system¹ if and only if $P \cdot [A + \Lambda_{k,k}] P^{-1} \in \mathbb{R}_+^{n \times n}$ and $P B Q^{-1} \in \mathbb{R}_+^{n \times m}$ for nonsingular matrix Q of the form*

$$Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in \mathbb{R}^{m \times m}$$

with i th, $i = 1, \dots, m$, rows given as $q_i = (q_{i1}, \dots, q_{im})$.

Let us define a feedback control law by

$$u(k) := \kappa(X(k)) \quad (14)$$

where $\kappa(x)$ is a vector function with values in \mathbb{R}^m . We say that a function $\kappa : \mathbb{R}^n \rightarrow J_0(m)$ is an *admissible feedback law* for system (8) if for every $k \in \mathbb{N}_0$ there exists a map $\gamma(X_0, \cdot, u)$ given by Eq. 11 such that $u(k) = \kappa(\gamma(X_0, a + kh, u(kh)))$. If κ is admissible feedback law for Eq. 8 (or respectively for Eqs. 7a or 7b), then the closed loop system is of the form

$$\left({}_{n_0} \Delta^{(\alpha)} X \right) (k) = \bar{F}(k, X(k), \kappa(X(k))), \quad (15)$$

where $X(k)$ is given by Eq. 10. As an immediate consequence of Theorem 10, we have the following.

Corollary 12 *Let K_P and \mathcal{P} be given as in Eqs. 12 and 13 together with $X_0 \in K_P$. If there exists a feedback law (14) such that for every $x \in K_P$, it holds*

$$\bar{F}(k, x, \kappa(X(k))) + \Lambda_{k,k} \cdot x \in K_P,$$

then for every $k \in \mathbb{N}_0$ system (15) is a \mathcal{P} cone system.

In particular, if system (8) is a linear one, i.e., $F(k, X(k), u(k)) = AX(k) + Bu(k)$, application of linear feedback law

$$\kappa = FX(k) \quad (16)$$

with a matrix $F \in \mathbb{R}^{m \times n}$ gives the closed loop system of the form

$$\left({}_{n_0} \Delta^{(\alpha)} X \right) (k) = (A + BF)X(k). \quad (17)$$

¹System (8) is called $(\mathcal{P}, \mathcal{Q})$ cone system if $X(\cdot) \in \mathcal{P}$ for every $X_0 \in K_P$ and every $u_i \in \mathcal{Q}$

4 Contractive Sets and Stability

Consider a polyhedron

$$\Omega(H, w) = \{x \in \mathbb{R}^n : Hx \leq w\} \quad (18)$$

where $H \in \mathbb{R}^{r \times n}$ and $w = (\omega_1, \dots, \omega_r)^T \in \mathbb{R}^r$ is a positively defined vector.

Definition 13 Polyhedron Ω given by Eq. 18 is λ -contractive set with respect to closed loop system (15) if there is $\lambda \in (0, 1]$ such that

$$\bar{F}(k, X(k), \kappa(X(k))) \in \Omega(H, w\varepsilon\lambda),$$

for all $\varepsilon \in (0, 1]$ and all $X(k) \in \Omega(H, w\varepsilon)$.

Recall that a set S is an invariant set for the system (15) if and only if every trajectory of this system starting within S remains inside it. Then λ -contractiveness with respect to system (15) implies invariance property for this system.

The one step admissible set to $\Omega(H, w)$ with respect to system (15) is defined by

$$\Gamma_1(H, w) := \{x \in \mathbb{R}^n : H\bar{\gamma}(X_0, a + h, u(h)) \leq w\}$$

The q step admissible set to $\Omega(H, w)$ with respect to system (15) is defined by

$$\Gamma_q(H, w) := \{x \in \mathbb{R}^n : H\bar{\gamma}(X_0, a + qh, u(qh)) \leq w\} \quad (19)$$

Proposition 14 Polyhedron Ω given by Eq. 18 is λ -contractive set with respect to closed loop system (15) if and only if there is $\lambda \in (0, 1]$ such that

$$\Omega(H, w\varepsilon) \subset \Gamma_1(H, w\varepsilon\lambda) \subset \dots \subset \Gamma_q(H, w\varepsilon\lambda)$$

for any $\varepsilon \in (0, 1]$ and natural q .

Proof The result is an immediate consequence of Definition 13 and of polyhedron (18). \square

Assumption 1 Let us assume that function \bar{F} given in Eq. 15 is continuous in all variables and (classically) continuously differentiable at the equilibrium point of the given system.

Under Assumption 1, let us define matrix

$$A := \frac{\partial \bar{F}}{\partial x}(X^e)$$

and consider a linear fractional order discrete-time system

$$({}_{n_0}\Delta^{(\alpha)}X)(k) = AX(k), \quad (20)$$

System (20) is called a *linear approximation* of the nonlinear one given by Eq. 15. Note that for a given initial condition and for an arbitrary sequence of controls $u \in J_0(m)$, there exists the unique solution of linear approximation (20). Recall that a constant vector $X^e = (0, \dots, 0)$ is an *equilibrium point* of fractional difference system (20) if and only if

$$({}_{n_0}\Delta^{(\alpha)}X^e)(k) = AX^e$$

for all $k \in \mathbb{N}_0$. Let us notice that the trivial solution $X \equiv 0$ is an equilibrium point of system (20). The equilibrium point $X^e = 0$ of Eq. 20 is said to be

- (a) *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|X_0\| < \delta$ implies $\|X(k)\| < \epsilon$, for all $k \in \mathbb{N}_0$;

(b) *attractive* if there exists $\delta > 0$ such that $\|X_0\| < \delta$ implies

$$\lim_{k \rightarrow \infty} X(k) = 0;$$

(iv) *asymptotically stable* if it is stable and attractive.

System (20) is called *stable/asymptotically stable* if its equilibrium point $X^e = 0$ is stable/asymptotically stable.

In the linear case, the effective characterization of asymptotic stability was given in [22].

Proposition 15 For $a \in \mathbb{R}$ and $\alpha \in (0, 1]$ let $\bar{x}(s) := x(sh)$. Then

$$\mathcal{Z}_{[n_0 \Delta^{(\alpha)} \bar{x}]}(z) = \left(\frac{hz}{z-1} \right)^{-\alpha_i} \left(Y_i(z) - \frac{z}{z-1} \right)^{-\beta_i} x_i(0)$$

where $X(z) = \mathcal{Z}[x](z)$ and $\beta_i = \alpha_i$, $i = 1, \dots, n$, for Riemann-Liouville-type operator and $\beta_i = 1$ for the Caputo type operator.

From Proposition 14, it follows that for any initial condition X_0 , the corresponding trajectories of system (20) are contained in $\Omega(H, w)$. The asymptotic stability of this system in polyhedron $\Omega(H, w)$ is guaranteed if all roots of the equation

$$\det(I - z^{-1} \mathcal{H} \Lambda A) = 0. \quad (21)$$

where A is the matrix given in Eq. 20, $\mathcal{H} := \text{diag}\{h^{\alpha_1}, \dots, h^{\alpha_p}\}$, $0 < \alpha_i \leq 1$, $\Lambda := \text{diag}\left\{\left(\frac{z}{z-1}\right)^{\alpha_1}, \dots, \left(\frac{z}{z-1}\right)^{\alpha_p}\right\}$ and z is a complex variable, are strictly inside the unit circle, see [22]. Then we have the following.

Proposition 16 Assume that there exists $\lambda \in (0, 1]$ and a feedback law $u(k) = FX(k)$ such that $\Omega(H, w)$ is λ -contractive set with respect to system (17). Then system (17) is locally asymptotically stable in $\Omega(H, w)$.

Proof Note that inside of polyhedron $\Omega(H, w)$, the origin is the only equilibrium point. Then the thesis follows from the compactness and from Proposition 14 and (21). \square

Proposition 17 Suppose that Assumption 1 is satisfied. Then there exists a neighborhood \mathcal{V} of the origin such that the closed loop system (17) is asymptotically stable.

Proof The result follows from Proposition 16 and from classical reasoning, see for example [9]. \square

Proposition 18 Let \bar{F} fulfill Assumption 1 and suppose there exist $\lambda \in (0, 1]$ and a feedback law $u(k) = FX(k)$ such that $\Omega(H, w\varepsilon\lambda)$, for every $X \in \Omega(H, w\varepsilon\lambda)$, is a non-empty λ -contractive set with respect to linearized system (20). Then there exists $\alpha \in (0, 1]$ such that set $\Omega(H, w\varepsilon\alpha)$, for every $X \in \Omega(H, w\varepsilon\alpha)$, is an invariant set for nonlinear system (15).

Proof We follow the reasoning from [9]. From Assumption 1 and the fact that $AX \in \Omega(H, w\varepsilon\lambda)$, for every $X \in \Omega(H, w\varepsilon\lambda)$, it is easy to infer that there exists $\alpha \in (0, 1]$ such

that $\bar{F}(k, X(k), \kappa(X(k))) \in \Omega(H, \omega\epsilon\alpha)$. This means that the set $\Omega(H, \omega\epsilon\alpha)$ is an invariant set for the nonlinear system (15). \square

5 Remarks on Asymptotic Controllability of Cone Systems

Let us draw our attention to the problem of controllability of system (8). Classically, controllability of the given system means that it is possible to transfer the considered system from a given initial state to a final state using controls from a certain set, see for example [16, 17].

Suppose that a set V is a subset of the state space of system (8).

Definition 19 Let $x_0, x_f \in V$. Then

- i. X is *asymptotically controlled to a final state* X_f without leaving V if there exists a control $u \in J_0(m)$ such that $\lim_{k \rightarrow \infty} \gamma(X_0, a + kh, u(kh)) = X_f$ and $\gamma(X_0, a + kh, u(kh)) \in V$ for all $k \in \mathbb{N}_0$.
- ii. If X^e is an equilibrium, then system (8) is *asymptotically controlled to* X^e if for each neighborhood V of X^e there is some neighborhood W of X^e such that each $X \in W$ can be asymptotically controlled to X^e without leaving V .

Proposition 20 Let K_P and \mathcal{P} be given as in Eqs. 12 and 13 together with $X_0 \in K_P$. Suppose that there exists some feedback law $u(k) = \kappa(X(k))$ so that X^e is a local asymptotically stable state for \mathcal{P} cone system (15). Then system (8) is asymptotically controlled to X^e .

Proof Since asymptotical stability of \mathcal{P} cone system (15) at X^e means that $\lim_{k \rightarrow \infty} \gamma(X_0, a + kh, \kappa(k)) = X^e$ and $\gamma(X_0, a + kh, \kappa(X(k)))$ is in a neighborhood of X^e for all $k \in \mathbb{N}_0$, hence the thesis follows from Definition 19. \square

Let us assume function F given in Eq. 8 fulfills Assumption 1.

Under this assumption, let us define matrices

$$A := \frac{\partial F}{\partial x}(X^e, u^e) \quad \text{and} \quad B := \frac{\partial F}{\partial u}(X^e, u^e)$$

and consider a linear fractional order discrete-time system

$$({}_{n_0}\Delta^{(\alpha)}X)(k) = AX(k) + Bu(k), \quad (22)$$

System (22) is called a *linear approximation* of the nonlinear one given by Eq. 8.

Proposition 21 Suppose there exist $\lambda \in (0, 1]$ and a feedback law $u(k) = \kappa(X(k))$ such that $\Omega(H, \omega\epsilon\lambda)$ is λ -contractive set with respect to closed loop system (20) and $\Omega(H, \omega\epsilon\lambda)$ is a compact polyhedron. Then linear system (22) is asymptotically controllable at X_0 .

Proof If there is law $u(k) = \kappa(X(k))$ such that $\Omega(H, \omega\epsilon\lambda)$ is λ -contractive set with respect to system (20) and $\Omega(H, \omega\epsilon\lambda)$ is a compact polyhedron, then by Proposition 17 the closed loop system (15) is asymptotically stable. Hence, by Definition 19, system (22) is asymptotically controllable at $X^e = X_0$. \square

6 Conclusions

We examined fractional discrete cone control systems with n -orders. Some relations between invariance and (asymptotic) stability properties of the presented systems were discussed. Since there are several definitions and notations of the fractional derivatives, among which the most popular are Caputo-, Riemman-Louville-, and Grünwald-Letnikov operators, we employ right them as fractional discrete Caputo-, Riemman-Louville-, and Grünwald-Letnikov type operators to the systems. In the paper, there were considered cone systems with control, which are particular invariant systems with control, together with their stability and asymptotic stability properties.

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